The Hopf algebra of dissection polylogarithms

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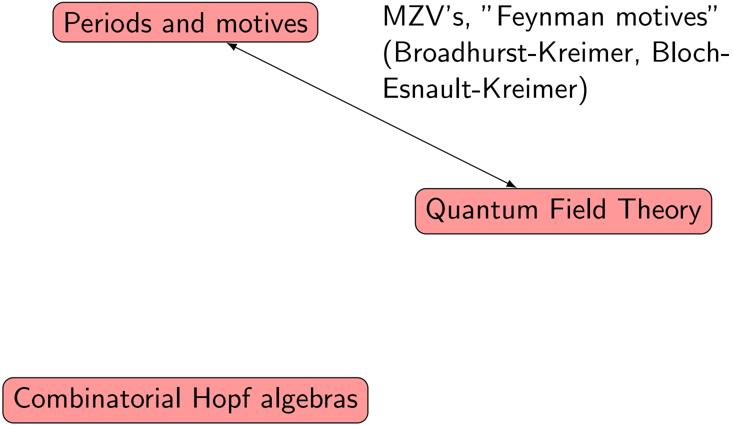
March 20, 2013

Periods and motives

Quantum Field Theory

Combinatorial Hopf algebras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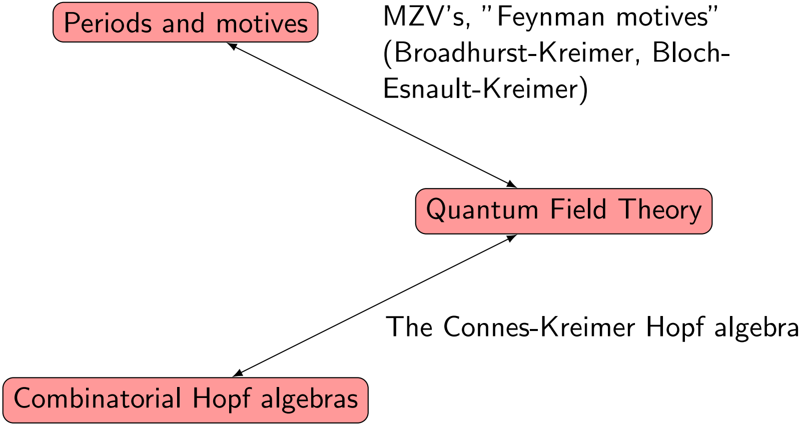
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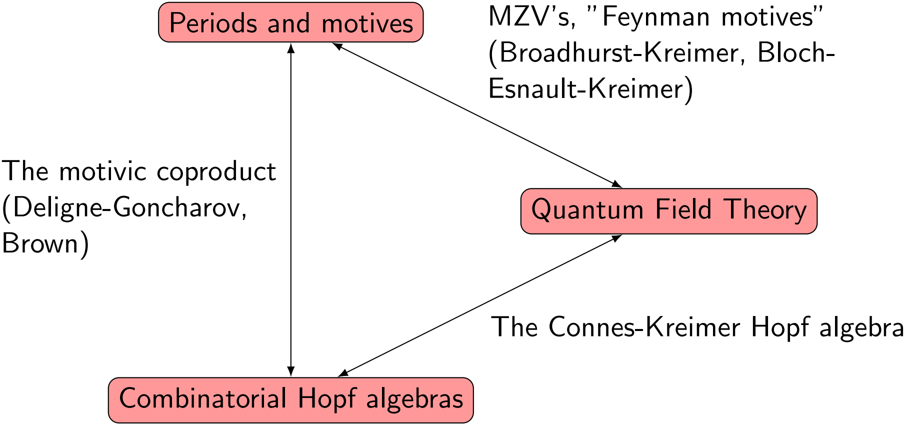
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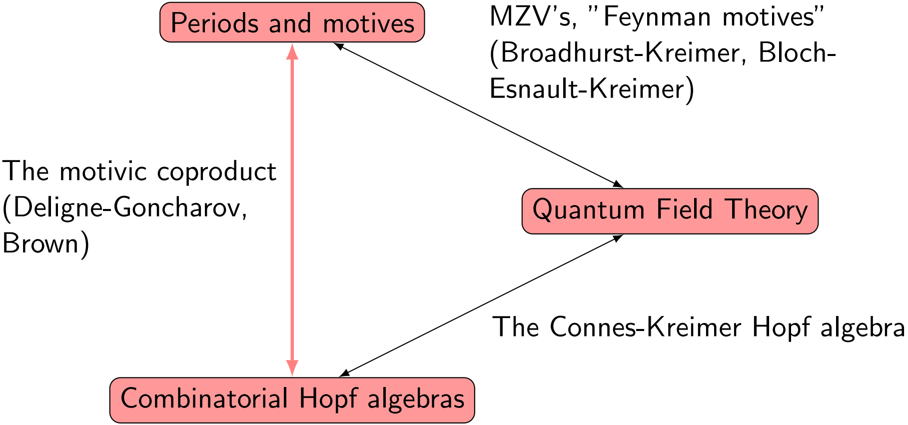
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Overview

We will define a graded Hopf algebra over $\mathbb{Q}$:

$$\mathcal{D} = \bigoplus_{n \geq 0} \mathcal{D}_n$$

which is

- connected: $\mathcal{D}_0 = \mathbb{Q}$
- commutative
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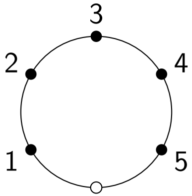
$\mathcal{D}$ is the free commutative algebra on the set of **dissection diagrams**.

Dissection diagrams



Dissection diagram of degree n

We start with a polygon with $n+1$ vertices, with a special vertex called the *root*.

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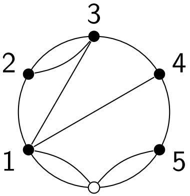

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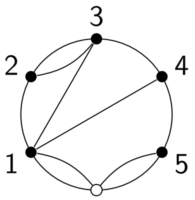

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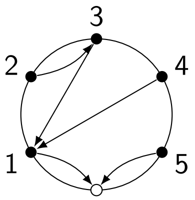

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Hence *the chords form a rooted tree*.

Examples

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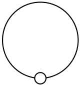
$\mathcal{D}_0 = \mathbb{Q} = \mathbb{Q}$ 

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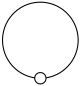
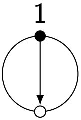
$\mathcal{D}_0 = \mathbb{Q} = \mathbb{Q}$ 

$\mathcal{D}_1 = \mathbb{Q}$ 

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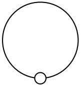
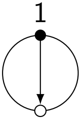
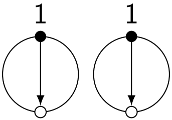
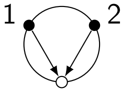
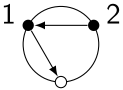
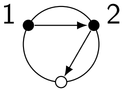
$\mathcal{D}_0 = \mathbb{Q} = \mathbb{Q}$ 

$\mathcal{D}_1 = \mathbb{Q}$ 

$\mathcal{D}_2 = \mathbb{Q}$ 

$\oplus\ \mathbb{Q}$ 

$\oplus\ \mathbb{Q}$ 

$\oplus\ \mathbb{Q}$ 

The coproduct

Definition

The coproduct $\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is defined, for D a dissection diagram, by the formula

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- D/S is obtained by contracting the chords from S $\rightsquigarrow$ product of dissection diagrams.

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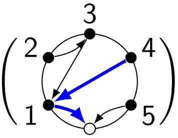
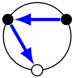
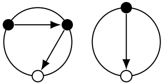
Δ 

contains the term  $\otimes$ 

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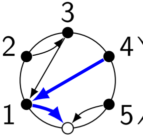
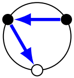
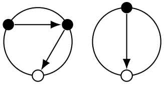
$\Delta\Bigg($ 

$\Bigg)$ contains the term  $\otimes$ 

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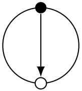
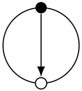
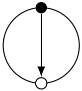
$\Delta\Bigg($  $\Bigg) = 1 \otimes$  $+$  $\otimes\ 1$

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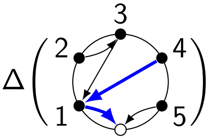

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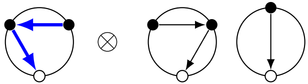

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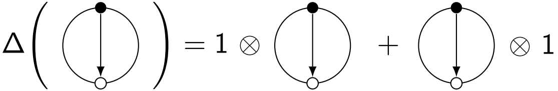

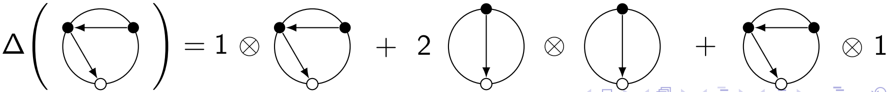

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We decorate the chords of the dissection with complex numbers $a_i \in \mathbb{C}$ and the edges of the polygons with complex numbers $b_j \in \mathbb{C}$.

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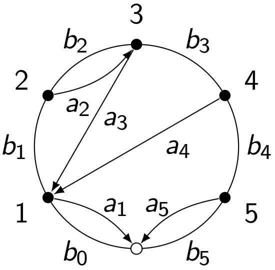

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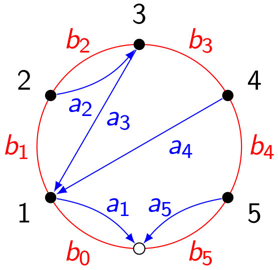

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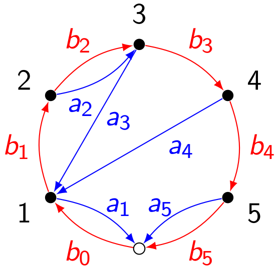

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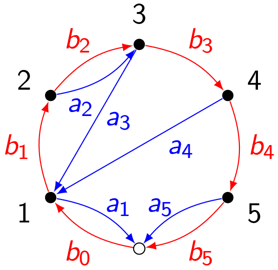

We get a graded Hopf algebra $\mathcal{D}^{dec}$ with a forgetful morphism $\mathcal{D}^{dec} \twoheadrightarrow \mathcal{D}$.

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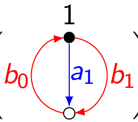
$I\Big($ 

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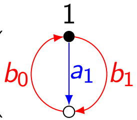
$I\Bigg($ 

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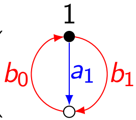
$I\Big($ 

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This genericity assumption will ensure that all the integrals $I(D)$ are convergent.

The differential form ω_D

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For a dissection diagram D of degree n, we set

$$\omega_D = \mathrm{dlog}(f_1) \wedge \cdots \wedge \mathrm{dlog}(f_n) = \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}$$

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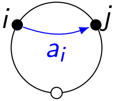

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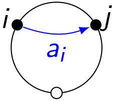

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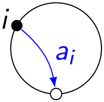

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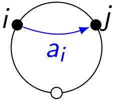

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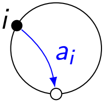

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We set $L_i = \{f_i = 0\} \subset \mathbb{C}^n$ hyperplane, so that $L = L_1 \cup \cdots \cup L_n$ is the divisor of the poles of ω_D.

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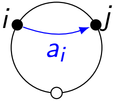

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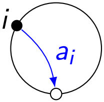

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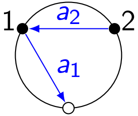
$D =$ 

$$\rightsquigarrow \quad \omega_D = \frac{dt_1 \wedge dt_2}{(t_1 - a_1)(t_2 - t_1 - a_2)}$$

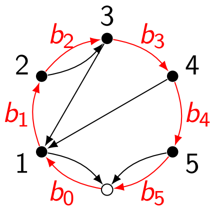
The integration simplex Δ_D

Definition

For a dissection diagram D of degree n, we define hyperplanes M_j in $\mathbb{C}^n$:

$$M_0 = \{0 = t_1 + b_0\}$$
$$M_j = \{t_j = t_{j+1} + b_j\} \text{ for } j = 1, \cdots, n-1$$
$$M_n = \{t_n = b_n\}$$

Let Δ_D be *any* topological simplex inside $\mathbb{C}^n \setminus L$ such that $\partial_j \Delta_D \subset M_j$ for all $j = 0, 1, \cdots, n$.



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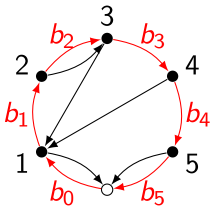

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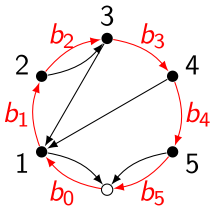

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The genericity assumption implies that $L \cup M$ is a normal crossing divisor inside $\mathbb{C}^n$, so that Δ_D always exists.

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The dissection polylogarithm attached to a dissection diagram D is the integral $I(D) = \int_{\Delta_D} \omega_D$ seen as a multi-valued function of the decorations a_i and b_j.

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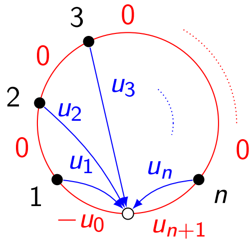
$D =$ 

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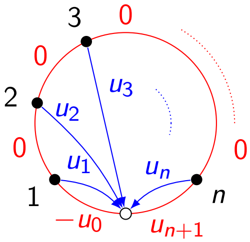
$D =$ 

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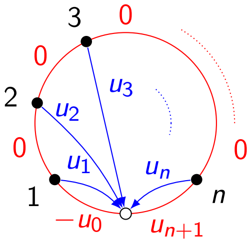
$D =$ 

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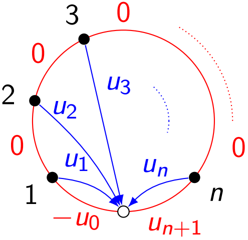

Non-generic example

$$\mathrm{Li}_2(t) = \int_{0 \leq x \leq y \leq t} \frac{dxdy}{(1-x)y} = -I(0; 1, 0; t)$$

Reduction to iterated integrals

Theorem (D.)

For every dissection diagram D of degree n, the dissection polylogarithm $I(D)$ is a linear combination with $\mathbb{Z}$-coefficients of (generic) iterated integrals $I(u_0; u_1, \cdots, u_n; u_{n+1})$ where the u_k's are linear combinations with $\mathbb{Z}$-coefficients of the decorations a_i and b_j of D.

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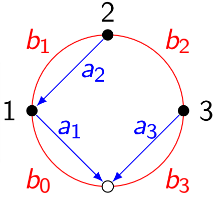

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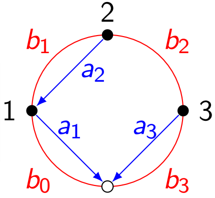

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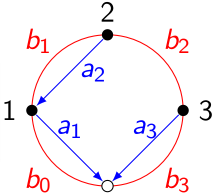

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- The number of iterated integrals that appear is between 1 and $n!$.

Overview

We will replace the dissection polylogarithms $I(D)$ by *motivic* versions

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The new feature is a *coproduct* $\Delta(I^{\mathcal{H}}(D)) = ?$

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Reconstructing the functions on G

If V is a representation of G, $v \in V$, $\varphi \in V^\vee$, then (V, v, φ) is a function on G:

$$g \mapsto \varphi(g.v)$$

Motivic dissection polylogarithms

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$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

If D has degree n then $I^{\mathcal{H}}(D) \in \mathcal{H}_n$

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$\Delta(I^{\mathcal{H}}(D)) \longleftrightarrow$ action of the motivic Galois group on $I(D)$.

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Theorem (D.)

The coproduct of the motivic dissection polylogarithms is given by the formula

$$\Delta(I^{\mathcal{H}}(D)) = \sum_{S \subset D} I^{\mathcal{H}}(S) \otimes I^{\mathcal{H}}(D/S)$$

In other words, the map

$$\mathcal{D}^{dec} \to \mathcal{H} \ , \ \ D \mapsto I^{\mathcal{H}}(D)$$

is a morphism of (graded) Hopf algebras.

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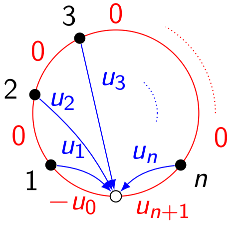

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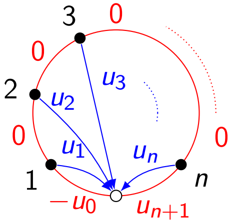

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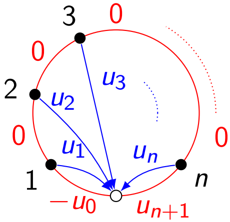

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Proposition (Goncharov, 2001)

$$\Delta(I^{\mathcal{H}}(u_0; u_1, \cdots, u_n; u_{n+1})) = \sum_{\substack{0 \leq k \leq n \\ 0=s_0<s_1<\cdots<s_k<s_{k+1}=n+1}}$$

$$I^{\mathcal{H}}(u_0; u_{s_1}, \cdots, u_{s_k}; u_{n+1}) \otimes \prod_{i=0}^{k} I^{\mathcal{H}}(u_{s_i}; u_{s_i+1}, \cdots, u_{s_{i+1}-1}; u_{s_{i+1}})$$

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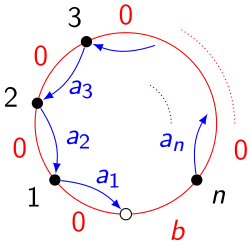

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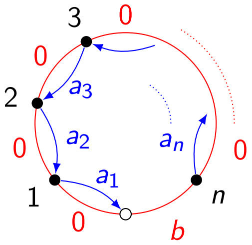

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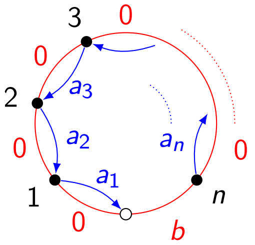

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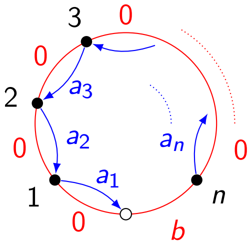

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Proposition

$$\Delta(J^{\mathcal{H}}(a_1,\cdots,a_n;b)) = \sum_{S\subset\{1,\cdots,n\}} J^{\mathcal{H}}(a(S);b)\otimes J^{\mathcal{H}}(a(\overline{S});b-a_S)$$

$$(a_S = \textstyle\sum_{s\in S} a_s)$$

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- $L_i = \{f_i = 0\} \subset \mathbb{C}^n$ hyperplanes $\rightsquigarrow$ $\omega_D = \mathrm{dlog}(f_1) \wedge \cdots \wedge \mathrm{dlog}(f_n)$.
- $M_j \subset \mathbb{C}^n$ hyperplanes $\rightsquigarrow$ Δ_D simplex such that $\partial_j \Delta_D \subset M_j$.

$$\rightsquigarrow I(D) = \int_{\Delta_D} \omega_D$$

Motivic dissection polylogarithm:

- $H := H^n(\mathbb{C}^n \setminus L, M \setminus M \cap L)$
- $\mathrm{gr}^W_{2n} H \cong H^n(\mathbb{C}^n \setminus L) \ni [\omega_D]$
- $(\mathrm{gr}^W_0 H)^\vee \cong H_n(\mathbb{C}^n, M) \ni [\Delta_D]$

$$\rightsquigarrow I^{\mathcal{H}}(D) = (H, [\omega_D], [\Delta_D]) \in \mathcal{H}$$

Theorem

$$\Delta(I^{\mathcal{H}}(D)) = \sum_{S \subset D} I^{\mathcal{H}}(S) \otimes I^{\mathcal{H}}(D/S)$$

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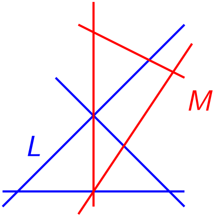

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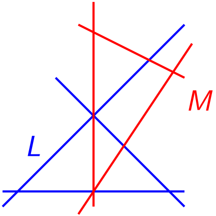

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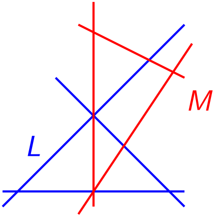

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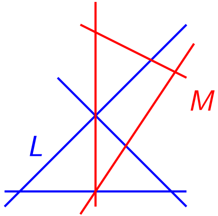

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What is known?

- The low-dimensional cases: $n \leq 3$.
- The generic case: $(L;M)$ in general position in $\mathbb{P}^n(\mathbb{C})$.
- The iterated integrals.

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The abstract formula for the coproduct

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Computing $\Delta(I^{\mathcal{H}}(L; M)) \longleftrightarrow$ finding functorial bases for $\mathrm{gr}^W_{2k} H(L; M)$.

A relative Brieskorn-Orlik-Solomon theorem

Theorem (Brieskorn-Orlik-Solomon)

Let $L = L_1 \cup \cdots \cup L_N$ be a union of linear hyperplanes in $\mathbb{C}^n$. Then we have an isomorphism of graded algebras

$$H^\bullet(\mathbb{C}^n \setminus L) \cong \Lambda^\bullet(e_1, \cdots, e_N)/\mathcal{R}$$

where $\mathcal{R}$ is generated by the relations:

$$\sum_{i=1}^k (-1)^i e_{s_1} \wedge \cdots \wedge \widehat{e_{s_i}} \wedge \cdots \wedge e_{s_k} = 0$$

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Theorem (D.)

For $k = 0, \cdots, n$ we have an explicit functorial presentation

$$\mathrm{gr}^W_{2k} H^n(\mathbb{P}^n(\mathbb{C}) \setminus L, M \setminus M \cap L) \cong \Lambda^k(e_0, \cdots, e_n) \otimes \Lambda^{n-k}(f_0, \cdots, f_n) \,/\, \mathcal{R}'$$